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Engineering Notes

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Expected Maneuver and Maneuver Covariance Models

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Introduction

WHEN performing a spacecraft maneuver, the actual force imparted by the propulsion subsystem is inevitably different from the force called for by the guidance and navigation software. Magnitude errors can be introduced by variability in thruster efficiency and the resolution of the timing in starting and stopping the thruster. Pointing errors can be introduced by misalignment of thrusters and spacecraft attitude errors. In turn, maneuver errors introduce error into the subsequent trajectory of the spacecraft. An accurate statistical model for the maneuver uncertainties can improve efficiency and help meet mission goals.

Discussion of the error statistics of spacecraft maneuvers is limited in the literature, but one relevant work is presented by Gates [1]. In that report, the author models burn errors from four sources. First, he establishes that errors occur in two directions: one in the direction of the intended burn vector and the other in a direction normal to the burn vector. In each of these directions, the errors are then divided into a portion that is proportional to the burn magnitude and a portion independent of the burn magnitude. He develops expressions for the second moment of the total error vector and concludes that in the absence of a priori information about the system, a spherical distribution for the error vector is appropriate.

This Note models the error as a magnitude error and an angular pointing error and then develops expressions for the expected value and covariance of the error vector associated with a planned spacecraft maneuver of imperfect magnitude and direction. It is shown that unbiased and independent magnitude and angle errors do not produce a burn with an expected value equal to the commanded burn, as intuition might suggest. Although the difference is small, precise maneuver planning for some applications may benefit from considering the effects of this bias.

Maneuver Model

The maneuver is modeled as an impulsive, instantaneous change to the velocity vector with no change in position. The commanded maneuver may be expressed in a suitable Cartesian inertial reference frame as a 3×1 vector, $\Delta \boldsymbol{V}_c$. For an Earth-orbiting spacecraft, an example of a suitable reference frame would be the Earth-centered-inertial frame, but this choice does not affect the following analysis. Similarly, the actual maneuver produced by the vehicle thrusters is designated as the 3×1 vector $\Delta \boldsymbol{V}_a$. The resultant error vector $\boldsymbol{e}$ is the difference

$$\boldsymbol{e} = \Delta \boldsymbol{V}_a - \Delta \boldsymbol{V}_c \tag{1}$$

In addition to the step change in the vehicle's velocity state caused by the burn, the preceding error also causes a step increase in the covariance of the state estimate. Because the burn is assumed to occur instantaneously, there is no translation of position during the course of the maneuver, and hence no position uncertainty is introduced [2]. Only the velocity covariance is increased. Through the application of the expectation operator $\mathrm{E}\{\cdot\}$ to the known burn statistics, analytic expressions are developed for the expected burn $\mathrm{E}\{\Delta \boldsymbol{V}_a\}$ and the burn covariance $P(\Delta \boldsymbol{V}_a)$, defined as

$$\mathrm{P}(\Delta \boldsymbol{V}_a) = \mathrm{E}\{[\Delta \boldsymbol{V}_a - \mathrm{E}\{\Delta \boldsymbol{V}_a\}][\Delta \boldsymbol{V}_a - \mathrm{E}\{\Delta \boldsymbol{V}_a\}]^T\} \tag{2}$$

This development is best accomplished in a reference frame with one axis, $\hat{\boldsymbol{X}}_V$, aligned with the intended burn direction and the other two axes orthogonal to $\hat{\boldsymbol{X}}_V$ but otherwise arbitrarily oriented, as shown in Fig. 1. This reference frame will be called the V-frame. In V-frame components, the commanded maneuver is

$$\Delta \boldsymbol{V}_c = \begin{bmatrix} \Delta V_{c,x} \\ \Delta V_{c,y} \\ \Delta V_{c,z} \end{bmatrix} = \begin{bmatrix} \Delta V_{c,x} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} |\Delta \boldsymbol{V}_c| \\ 0 \\ 0 \end{bmatrix} = \Delta V_{c,x} \hat{\boldsymbol{X}}_V + 0 \cdot \hat{\boldsymbol{Y}}_V + 0 \cdot \hat{\boldsymbol{Z}}_V \tag{3}$$

Error Sources

This Note assumes that the actual burn performed by the rocket motor includes two sources of error. The magnitude of the maneuver produced by the motor will not perfectly match the desired magnitude, and the direction will not be perfect. The magnitude error is denoted as M, and the pointing error is described by two random variables denoted as θ and ϕ. The orientation of the actual burn relative to the V-frame is shown in Fig. 1.

In the figure, the solid arc represents the length of the desired $\Delta \boldsymbol{V}_c$ vector, $|\Delta \boldsymbol{V}_c|$. In the example shown, the actual burn is larger than the commanded burn, so that $M > 0$ and $|\Delta \boldsymbol{V}_a| > |\Delta \boldsymbol{V}_c|$.

The magnitude error M is modeled as a zero-mean random variable with a normal distribution and a variance of σ_M^2. Modeling the direction distribution is less straightforward, due to the angular nature of the variables. One choice is to use a von Mises–Fisher joint distribution for the two angles. Another alternative is to use a uniform distribution for the azimuth angle ϕ combined with a wrapped normal distribution for the off-axis angle θ. However, under the assumption that the variance of the off-axis variable θ will be much smaller than π, both choices can be reasonably approximated using a normal distribution for θ and a uniform distribution for ϕ [3]. The off-axis error θ is assumed to have zero mean and a variance of σ_θ^2. The azimuth angle ϕ is a random variable with a uniform distribution between 0 and π. Because θ may be positive or negative, this range of

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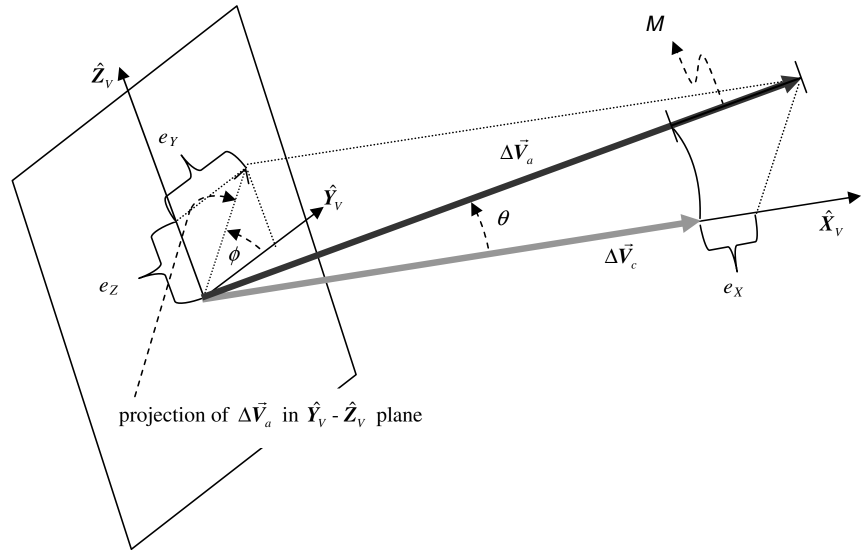

Fig. 1 Maneuver geometry and error sources.

ϕ is chosen so that any direction has a one-to-one correspondence with an angle pair (ϕ, θ). The three random variables are assumed independent. Using this model, the expected correlations E$\{M\ \theta\}$, E$\{M\ \phi\}$, and E$\{\theta\ \phi\}$ are all zero.

The V-frame components of the error vector $\boldsymbol{e}$ are denoted by e_X, e_Y, and e_Z, as shown in Fig. 1. The distribution of these error components can be determined using the known distributions of the underlying random variables M, θ and ϕ. Abbreviating the magnitude of the commanded burn vector $|\Delta \boldsymbol{V}_c|$ as ΔV_c, the error components can be written in terms of the underlying error sources and the known ΔV_c as

$$\begin{bmatrix} e_X \\ e_Y \\ e_Z \end{bmatrix} = \begin{bmatrix} (\Delta V_c + M)\cos\theta - \Delta V_c \\ (\Delta V_c + M)\sin\theta\cos\phi \\ (\Delta V_c + M)\sin\theta\sin\phi \end{bmatrix} \tag{4}$$

Expected Burn

Because the commanded maneuver is deterministic, the expected value of the commanded burn is

$$\mathrm{E}\{\Delta \boldsymbol{V}_c\} = \Delta \boldsymbol{V}_c \tag{5}$$

Using the linearity of the expectation operator, the expected value of the actual burn is

$$\mathrm{E}\{\Delta \boldsymbol{V}_a\} = \mathrm{E}\left\{\Delta \boldsymbol{V}_c + \begin{bmatrix} e_X \\ e_Y \\ e_Z \end{bmatrix}\right\} = \Delta \boldsymbol{V}_c + \mathrm{E}\left\{\begin{bmatrix} e_X \\ e_Y \\ e_Z \end{bmatrix}\right\} \tag{6}$$

Combining Eqs. (4–6) gives

$$\mathrm{E}\{\Delta \boldsymbol{V}_a\} = \Delta \boldsymbol{V}_a + \begin{bmatrix} \Delta V_c \mathrm{E}\{\cos\theta\} + \mathrm{E}\{M\cos\theta\} - \Delta V_c \\ \Delta V_c \mathrm{E}\{\sin\theta\cos\phi\} + \mathrm{E}\{M\sin\theta\cos\phi\} \\ \Delta V_c \mathrm{E}\{\sin\theta\sin\phi\} + \mathrm{E}\{M\sin\theta\sin\phi\} \end{bmatrix} \tag{7}$$

To evaluate the expected value operations in Eq. (7), the probability density functions for the random variables are required. Based on the distributions chosen previously, these functions are

$$p_M(M) = \frac{1}{\sqrt{2\pi\sigma_M^2}} \exp\left(\frac{-M^2}{2\sigma_M^2}\right), \qquad -\infty < M < \infty \tag{8}$$

$$p_\theta(\theta) = \frac{1}{\sqrt{2\ \pi\ \sigma_\theta^2}} \exp\left(\frac{-\theta^2}{2\ \sigma_\theta^2}\right), \qquad -\infty < \theta < \infty \tag{9}$$

$$p_\phi(\phi) = \frac{1}{\pi}, \qquad 0 \le \phi \le \pi \qquad p_\phi(\phi) = 0, \qquad \text{elsewhere} \tag{10}$$

In Eqs. (8) and (9), σ_M and σ_θ are the standard deviations of the magnitude and off-axis angle, respectively. Applying the definition of the expectation operator to the first component of the error term in Eq. (7) yields

$$\begin{aligned} \mathrm{E}\{e_X\} &= \Delta V_c \int_{-\infty}^{\infty} \cos\theta p_\theta(\theta)\,\mathrm{d}\theta - \Delta V_c \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} M\cos\theta p_M(M) p_\theta(\theta)\,\mathrm{d}\theta\,\mathrm{d}M \\ &= \Delta V_c \int_{-\infty}^{\infty} \cos\theta p_\theta(\theta)\,\mathrm{d}\theta - \Delta V_c \\ &+ \int_{-\infty}^{\infty} M p_M(M)\,\mathrm{d}M \int_{-\infty}^{\infty} \cos\theta p_\theta(\theta)\,\mathrm{d}\theta \end{aligned} \tag{11}$$

where the independence of the error sources yields a form of integral that can be separated into factors. The integrals in Eq. (11) (along with similar integrals in the rest of this Note) may be evaluated using an integral table [4]:

$$\int_{-\infty}^{\infty} \cos\theta p_\theta(\theta)\,\mathrm{d}\theta = e^{-\frac{1}{2}\sigma_\theta^2} \tag{12}$$

$$\int_{-\infty}^{\infty} M p_M(M)\,\mathrm{d}M = 0 \tag{13}$$

Substituting Eqs. (12) and (13) into Eq. (11), the expected value of the first component of the error simplifies to

$$\mathrm{E}\{e_X\} = \Delta V_c e^{-\frac{1}{2}\sigma_\theta^2} - \Delta V_c \tag{14}$$

Combining this with Eq. (6) and using $\Delta V_{c,x} = \Delta V_c$ gives

$$\mathrm{E}\{\Delta V_{a,x}\} = \Delta V_c e^{-\frac{1}{2}\sigma_\theta^2} \tag{15}$$

The next component of the error term in Eq. (7) is

$$\begin{aligned}E\{e_Y\} &= \Delta V_c E\{\sin\theta\cos\phi\} + E\{M\sin\theta\cos\phi\}\\ &= \Delta V_c \int_{-\infty}^{\infty}\int_0^{\pi} \sin\theta\cos\phi\, p_\theta(\theta)\, p_\phi(\phi)\,d\phi\,d\theta\\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\pi} M\sin\theta\cos\phi\, p_M(M) p_\theta(\theta)\, p_\phi(\phi)\,d\phi\,d\theta\,dM\\ &= \Delta V_c \int_{-\infty}^{\infty}\sin\theta p_\theta(\theta)\,d\theta \int_0^{\pi}\cos\phi p_\phi(\phi)\,d\phi\\ &+ \int_{-\infty}^{\infty} M\, p_M(M)\,dM \int_{-\infty}^{\infty}\sin\theta\, p_\theta(\theta)\,d\theta \int_0^{\pi}\cos\phi p_\phi(\phi)\,d\phi\end{aligned} \tag{16}$$

Note that both terms in Eq. (16) contain the same integral involving the azimuth angle ϕ, which evaluates to zero,

$$\int_0^{\pi}\cos\phi p_\phi(\phi)\,d\phi = 0 \tag{17}$$

so that the righthand side of Eq. (16) may be simplified to zero, and the second component of Eq. (16) can be shown to be

$$E\{\Delta V_{a,y}\} = 0 \tag{18}$$

Similarly it can be shown that the third component of Eq. (6) is also zero,

$$E\{\Delta V_{a,z}\} = 0 \tag{19}$$

Using Eqs. (15), (18), and (19), the complete estimated burn may now be written in terms of the commanded burn and the error statistics, as

$$E\{\Delta \boldsymbol{V}_a\} = \begin{bmatrix}\Delta V_c e^{-\frac{1}{2}\sigma_\theta^2}\\ 0\\ 0\end{bmatrix} \tag{20}$$

This result is significant because it indicates that the expected error in the in-track direction is not symmetric about zero. If there is any uncertainty in the spacecraft attitude, the expected cross-track errors will be zero (first moments), but the in-track error will have a nonzero mean: the burn will fall short more often than not. This negative bias is very small unless the pointing error is very large: the magnitude bias does not exceed 1% until the standard deviation of the angular error is nearly 8 deg. However, for burns of finite duration this may become significant and might be taken into account for rotating boosters, for which some coning motion is inevitable. To account for this effect, the commanded burn would need a slightly larger magnitude than that of the desired maneuver.

Covariance Matrix

To determine the burn covariance, start with the difference between the actual burn vector and its expected value:

$$\begin{aligned}\Delta \boldsymbol{V}_a - E\{\Delta \boldsymbol{V}_a\} &= \left(\begin{bmatrix}\Delta V_c\\ 0\\ 0\end{bmatrix} + \begin{bmatrix}e_X\\ e_Y\\ e_Z\end{bmatrix}\right) - \begin{bmatrix}\Delta V_c e^{-\frac{1}{2}\sigma_\theta^2}\\ 0\\ 0\end{bmatrix}\\ &= \begin{bmatrix}e_X + \Delta V_c - \Delta V_c e^{-\frac{1}{2}\sigma_\theta^2}\\ e_Y\\ e_Z\end{bmatrix}\end{aligned} \tag{21}$$

Now the covariance of the actual burn is given by

$$P(\Delta V_a) = E\{[\Delta \boldsymbol{V}_a - E\{\Delta \boldsymbol{V}_a\}][\Delta \boldsymbol{V}_a - E\{\Delta \boldsymbol{V}_a\}]^T\} \tag{22}$$

Substituting Eq. (21) into Eq. (22) and expanding, the nine components of the covariance matrix can be written as

$$\begin{aligned}&P_{(1,1)}(\Delta \boldsymbol{V}_a) = E\{(e_X + \Delta V_c - \Delta V_c e^{-\frac{1}{2}\sigma_\theta^2})^2\}\\ &P_{(1,2)}(\Delta \boldsymbol{V}_a) = P_{(2,1)}(\Delta \boldsymbol{V}_a) = E\{e_X e_Y + e_Y\Delta V_c - e_Y \Delta V_c e^{-\frac{1}{2}\sigma_\theta^2}\}\\ &P_{(1,3)}(\Delta \boldsymbol{V}_a) = P_{(3,1)}(\Delta \boldsymbol{V}_a) = E\{e_X e_Z + e_Z\Delta V_c - e_Z \Delta V_c e^{-\frac{1}{2}\sigma_\theta^2}\}\\ &P_{(2,2)}(\Delta \boldsymbol{V}_a) = E\{e_Y^2\} \qquad P_{(2,3)}(\Delta \boldsymbol{V}_a) = P_{(3,2)}(\Delta \boldsymbol{V}_a) = E\{e_Y e_Z\}\\ &P_{(3,3)}(\Delta \boldsymbol{V}_a) = E\{e_Z^2\}\end{aligned} \tag{23}$$

Beginning with the first term,

$$\begin{aligned}E\{(e_X + \Delta V_c - \Delta V_c e^{-\frac{1}{2}\sigma_\theta^2})^2\} &= E\{\Delta V_c^2\cos^2\theta\} + E\{M^2\cos^2\theta\}\\ &+ E\{\Delta V_c^2 e^{-\sigma_\theta^2}\} + E\{2\Delta V_c M\cos^2\theta\} - E\{2\Delta V_c^2\cos\theta e^{-\frac{1}{2}\sigma_\theta^2}\}\\ &- E\{2\Delta V_c M\cos\theta e^{-\frac{1}{2}\sigma_\theta^2}\}\end{aligned} \tag{24}$$

Again, the deterministic nature of ΔV_c, ΔV_c^2, and σ_θ^2 and the independence of the error sources allows us to write this as

$$\begin{aligned}E\{(e_X + \Delta V_c - \Delta V_c e^{-\frac{1}{2}\sigma_\theta^2})^2\} &= \Delta V_c^2 \int_{-\infty}^{\infty}\cos^2\theta p_\theta(\theta)\,d\theta\\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} M^2\cos^2\theta p_M(M)\, p_\theta(\theta)\,d\theta\,dM + \Delta V_c^2 e^{-\sigma_\theta^2}\\ &+ 2\Delta V_c \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} M\cos^2\theta p_M(M)\, p_\theta(\theta)\,d\theta\,dM\\ &- 2\Delta V_c^2 e^{-\frac{1}{2}\sigma_\theta^2}\int_{-\infty}^{\infty}\cos\theta p_\theta(\theta)\,d\theta\\ &- 2\Delta V_c e^{-\frac{1}{2}\sigma_\theta^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} M\cos\theta p_M(M)\, p_\theta(\theta)\,d\theta\,dM\end{aligned} \tag{25}$$

As with the analysis of the expected values, the independence of the random variables allows the separation of integrals, which can then be evaluated as

$$\int_{-\infty}^{\infty}\cos^2\theta\, p_\theta(\theta)\,d\theta = \frac{1}{2}(1 + e^{-2\sigma_\theta^2}) \tag{26}$$

and

$$\int_{-\infty}^{\infty} M^2 p_M(M)\,dM = \sigma_M^2 \tag{27}$$

Then the (1,1) component of the covariance matrix reduces to

$$P_{(1,1)}(\Delta \boldsymbol{V}_a) = \tfrac{1}{2}(\Delta V_c^2 + \sigma_M^2)(1 + e^{-2\sigma_\theta^2}) - \Delta V_c^2 e^{-\sigma_\theta^2} \tag{28}$$

This is one of the nine covariance matrix components. It can be shown that all of the off-diagonal components vanish either as a result of Eq. (17) or of the integral

$$\int_{-\infty}^{\infty}\sin\theta p_\theta(\theta)\,d\theta = 0 \tag{29}$$

The (2,2) component is given by

$$\begin{aligned}E\{e_Y^2\} &= E\{[(\Delta V_c + M)\sin\theta\cos\phi]^2\} = \Delta V_c^2 E\{\sin^2\theta\cos^2\phi\}\\ &+ 2\Delta V_c E\{M\sin^2\theta\cos^2\phi\} + E\{M^2\sin^2\theta\cos^2\phi\}\end{aligned} \tag{30}$$

In the application of the expectation operation, the integrals that must be evaluated are

$$\int_{-\infty}^{\infty}\sin^2\theta p_\theta(\theta)\,d\theta = \frac{1}{2}(1 - e^{-2\sigma_\theta^2}) \tag{31}$$

and

$$\int_0^{\pi}\cos^2\phi p_\phi(\phi)\,d\phi = \frac{1}{2} \tag{32}$$

With these integrals, Eq. (30) can be evaluated as

$$P_{(2,2)}(\Delta \boldsymbol{V}_a) = \mathrm{E}\{e_Y^2\} = \tfrac{1}{4}(\Delta V_c^2 + \sigma_M^2)(1 - e^{-2\sigma_\theta^2}) \tag{33}$$

Finally, component (3,3) is given by

$$\mathrm{E}\{e_Z^2\} = \mathrm{E}\{[(\Delta V_c + M)\sin\theta\sin\phi]^2\} = \Delta V_c^2 \mathrm{E}\{\sin^2\theta\sin^2\phi\} + 2\Delta V_c \mathrm{E}\{M\sin^2\theta\sin^2\phi\} + \mathrm{E}\{M^2\sin^2\theta\sin^2\phi\} \tag{34}$$

One more useful result from the integral table is

$$\int_0^\pi \sin^2\phi\, p_\phi(\phi)\,\mathrm{d}\phi = \frac{1}{2} \tag{35}$$

With this result, Eq. (34) can be evaluated as

$$P_{(3,3)}(\Delta \boldsymbol{V}_a) = \mathrm{E}\{e_Z^2\} = \tfrac{1}{4}(\Delta V_c^2 + \sigma_M^2)(1 - e^{-2\sigma_\theta^2}) \tag{36}$$

Comparing Eqs. (33) and (36), it is evident that the expressions are equivalent. We can now write out the complete covariance matrix:

$$P(\Delta \boldsymbol{V}_a) = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & B \end{bmatrix} \tag{37}$$

where A is defined by Eq. (28), and B is defined by either Eq. (33) or Eq. (36). Because the covariances in the second and third axes are identical, the alignment of the V-frame relative to the actual error (the value of ϕ) does not matter. Also, we note that the matrix is diagonal. This indicates no correlation of error among the three axes. Note that this was developed here, not assumed a priori.

Covariance Approximation

In many instances, it will be reasonable to assume that the variances of the error sources (σ_M^2, σ_θ^2, and σ_ϕ^2) will be small. In this case, the expression for the covariance given in Eq. (37) can be approximated by a much simpler expression. Under the assumption that $\sigma_\theta^2 \ll 1$, the exponential terms may be approximated by the first two terms of the series expansion for the exponential function (i.e., $e^x \approx 1 + x$ for small x). In addition, if $\sigma_M^2 \ll \Delta V_c^2$, then the approximation $\Delta V_c^2 + \sigma_M^2 \approx \Delta V_c^2$ is reasonable. Using these simplifications, the covariance is approximately

$$P(\Delta \boldsymbol{V}_a) \approx \begin{bmatrix} \sigma_M^2 & 0 & 0 \\ 0 & \frac{1}{2}\Delta V_c^2\sigma_\theta^2 & 0 \\ 0 & 0 & \frac{1}{2}\Delta V_c^2\sigma_\theta^2 \end{bmatrix} \tag{38}$$

This approximation would be accurate enough for most mission-planning applications and probably sufficient for guidance algorithms. As an example, consider a commanded maneuver of $\Delta V_c = 100$ m/s, with error standard deviations of $\sigma_M = 5$ m/s and $\sigma_\theta = 0.02$ rad. In this case, Eq. (37) gives

$$P(\Delta \boldsymbol{V}_a) = \begin{bmatrix} 24.99 & 0 & 0 \\ 0 & 2.004 & 0 \\ 0 & 0 & 2.004 \end{bmatrix} \mathrm{m^2/s^2} \tag{39}$$

and the approximation of Eq. (38) gives

$$P(\Delta \boldsymbol{V}_a) \approx \begin{bmatrix} 25.00 & 0 & 0 \\ 0 & 2.000 & 0 \\ 0 & 0 & 2.000 \end{bmatrix} \mathrm{m/s^2} \tag{40}$$

In this case, the approximation given by Eq. (38) results in errors less than 0.1% of the value predicted by Eq. (37).

Desired Burn

As noted earlier with Eq. (20), the expected burn will not equal the commanded burn. If it is essential to minimize the effect of burn errors, it may be useful to distinguish between a desired burn and the commanded burn. That is, we intentionally command a burn slightly different from the desired burn. To achieve a desired burn, defined as

$$\Delta \boldsymbol{V}_d = \begin{bmatrix} \Delta V_d \\ 0 \\ 0 \end{bmatrix} \tag{41}$$

we can set $\mathrm{E}\{\Delta \boldsymbol{V}_a\} = \Delta \boldsymbol{V}_a$ by commanding the following burn:

$$\Delta \boldsymbol{V}_c = \begin{bmatrix} \Delta V_d e^{\frac{1}{2}\sigma_\theta^2} \\ 0 \\ 0 \end{bmatrix} \tag{42}$$

With this approach, the expected burn is the desired burn, and the covariance takes the following form:

$$P(\Delta \boldsymbol{V}_a) = \begin{bmatrix} A' & 0 & 0 \\ 0 & B' & 0 \\ 0 & 0 & B' \end{bmatrix} \tag{43}$$

where

$$A' = \tfrac{1}{2}\Delta V_d^2(e^{\sigma_\theta^2} + e^{-\sigma_\theta^2}) + \tfrac{1}{2}\sigma_M^2(1 + e^{-2\sigma_\theta^2}) - \Delta V_d^2 \tag{44}$$

$$B' = \tfrac{1}{4}\Delta V_d^2(e^{\sigma_\theta^2} - e^{-\sigma_\theta^2}) + \tfrac{1}{4}\sigma_M^2(1 - e^{-2\sigma_\theta^2}) \tag{45}$$

This approach shifts the expected error ellipsoid associated with the maneuver such that the center of the ellipsoid lies precisely at the desired maneuver.

Conclusions

This Note has developed the expected burn and burn covariance associated with a planned, impulsive satellite maneuver with normally distributed errors in magnitude and direction. It has shown that in the case of imperfect pointing, the expected maneuver does not equal the commanded maneuver: it is slightly biased. This bias grows very slowly with the pointing error variance, but for cases in which the variance is large, a method was presented to determine a maneuver command for which the expected result matches the desired result. Most significantly, this Note has rigorously shown that the approximation given by Eq. (38) is justified for use in most mission-planning applications and provides a means of bounding the errors introduced by their use.

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